

A SMALL IMPROVEMENT IN THE SMALL GAPS BETWEEN CONSECUTIVE ZEROS OF THE RIEMANN ZETA-FUNCTION¹

SERGEI PREOBRAZHENSKIĬ

ABSTRACT. Feng and Wu introduced a new general coefficient sequence into Montgomery and Odlyzko's method for exhibiting irregularity in the gaps between consecutive zeros of $\zeta(s)$ assuming the Riemann Hypothesis. They used a special case of their sequence to improve upon earlier results on the gaps. In this paper we consider an equivalent form of the general sequence of Feng and Wu, and introduce a somewhat less general sequence $\{a_n\}$ for which we write the Montgomery–Odlyzko expressions explicitly. As an application, we give the following slight improvement of Feng and Wu's result: infinitely often consecutive non-trivial zeros of the Riemann zeta-function differ by at most 0.515396 times the average spacing.

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1. Introduction. It is well known that the Riemann zeta-function $\zeta(s)$ has infinitely many nontrivial zeros $s = \rho = \beta + i\gamma$, and all of them are in the critical strip $0 < \operatorname{Re} s = \sigma < 1$, $-\infty < \operatorname{Im} s = t < \infty$.

If $N(T)$ denotes the number of zeros $\rho = \beta + i\gamma$ (β and γ real), for which $0 < \gamma \leq T$, then

$$N(T) = \frac{T}{2\pi} \log \left(\frac{T}{2\pi} \right) - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O \left(\frac{1}{T} \right),$$

with

$$S(T) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right)$$

and

$$S(T) = O(\log T).$$

This is the Riemann–von Mangoldt formula for $N(T)$. Hence, if we let $0 < \gamma \leq \gamma'$ denote consecutive ordinates of non-trivial zeros of $\zeta(s)$, the average size of $\gamma' - \gamma$ is $\gamma/N(\gamma) \sim 2\pi/\log \gamma$. Let

$$\lambda = \limsup_{\gamma > 0} (\gamma' - \gamma) \frac{\log \gamma}{2\pi}$$

and

$$\mu = \liminf_{\gamma > 0} (\gamma' - \gamma) \frac{\log \gamma}{2\pi}.$$

We note that $\mu \leq 1 \leq \lambda$ and it is expected that $\mu = 0$ and $\lambda = +\infty$.

Let $N_0(T)$ be the number of zeros of $\zeta \left(\frac{1}{2} + it \right)$ when $0 < t \leq T$, each zero counted with multiplicity. The Riemann hypothesis is the conjecture that $N_0(T) = N(T)$.

In this note we prove the following theorem.

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Theorem 1. *Assume the Riemann Hypothesis. Then we have*

$$\mu < 0.515396.$$

We briefly describe the history of the problem, focusing mainly on μ .

- [7]: in 1946 Selberg remarked that $\mu < 1 < \lambda$ unconditionally.

Now suppose that T is a large real number and $K = T(\log T)^{-2}$. Let

$$h(c) = c - \frac{\operatorname{Re} \left(\sum_{n \leq K} a_n \overline{a_{nk}} g_c(n) \Lambda(n) n^{-1/2} \right)}{\sum_{k \leq K} |a_k|^2} \quad (1)$$

where

$$g_c(n) = \frac{2 \sin \left(\pi c \frac{\log n}{\log T} \right)}{\pi \log n}$$

and Λ is the von Mangoldt's function.

In the following results, the truth of the Riemann Hypothesis is assumed.

- [6]: in 1981 by an argument using the Guinand–Weil explicit formula, Montgomery and Odlyzko showed that if $h(c) < 1$ for some choice of c and $\{a_n\}$, then $\lambda \geq c$, and if $h(c) > 1$ for some choice of c and $\{a_n\}$, then $\mu \leq c$. They used the coefficients

$$a_k = \frac{1}{k^{\frac{1}{2}}} f \left(\frac{\log k}{\log K} \right) \quad \text{and} \quad a_k = \frac{\lambda(k)}{k^{\frac{1}{2}}} f \left(\frac{\log k}{\log K} \right)$$

where f is a continuous function of bounded variation, and $\lambda(k)$ is the Liouville function. With this choice of the coefficients they obtained $\lambda > 1.9799$ and $\mu < 0.5179$ by optimizing over the functions f .

- [2]: in 1984 Conrey, Ghosh & Gonek chose the coefficients

$$a_k = \frac{d_r(k)}{\sqrt{k}} \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}}$$

where $d_r(k)$ is a multiplicative function defined on integral powers of a prime p by

$$d_r(p^k) = \frac{\Gamma(k+r)}{\Gamma(r)k!}.$$

The choice $r = 1.1$ with the latter a_k yields $\mu < 0.5172$ and the choice $r = 2.2$ with the former a_k yields $\lambda > 2.337$.

- [4]: in 2005, by making use of the Wirtinger's inequality and the asymptotic formulae for the fourth mixed moments of the zeta-function and its derivative, R. R. Hall proved that $\lambda > 2.6306$.
- [1]: in 2010, Bui, Milinovich & Ng considered the coefficients of the form

$$a_k = \frac{d_r(k)}{\sqrt{k}} f \left(\frac{\log K/k}{\log K} \right) \quad \text{and} \quad a_k = \frac{\lambda(k) d_r(k)}{\sqrt{k}} f \left(\frac{\log K/k}{\log K} \right)$$

for a polynomial f and obtained $\lambda > 2.69$ and $\mu < 0.5155$.

- [3]: in 2012, Feng & Wu introduced the coefficient

$$a_k = \frac{d_r(k)}{k^{\frac{1}{2}}} \left(f_1 \left(\frac{\log K/k}{\log K} \right) + f_2 \left(\frac{\log K/k}{\log K} \right) \sum_{p_1 p_2 | k} \frac{\log p_1 \log p_2}{\log^2 K} \right. \\ \left. + f_3 \left(\frac{\log K/k}{\log K} \right) \sum_{p_1 p_2 p_3 | k} \frac{\log p_1 \log p_2 \log p_3}{\log^3 K} + \dots \right. \\ \left. + f_I \left(\frac{\log K/k}{\log K} \right) \sum_{p_1 p_2 \dots p_I | k} \frac{\log p_1 \log p_2 \dots \log p_I}{\log^I K} \right),$$

for any integer $I \geq 2$. Using $I = 2$ they obtained $\lambda > 2.7327$ and $\mu < 0.5154$, or, to higher precision, $\mu < 0.515398$.

We remark that the coefficient of Feng & Wu is equivalent to the coefficient

$$a_k = \frac{d_r(k)}{k^{\frac{1}{2}}} \left(f_1 \left(\frac{\log K/k}{\log K} \right) + f_2 \left(\frac{\log K/k}{\log K} \right) \sum_{p_1 | k} \frac{\log^2 p_1}{\log^2 K} \right. \\ \left. + f_3 \left(\frac{\log K/k}{\log K} \right) \sum_{p_1 | k} \frac{\log^3 p_1}{\log^3 K} + \dots \right. \\ \left. + f_I \left(\frac{\log K/k}{\log K} \right) \sum_{p_1 | k} \frac{\log^I p_1}{\log^I K} \right),$$

for which the calculations are simpler.

To prove Theorem 1, we choose the coefficients

$$a_k = \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} f_1 \left(\frac{\log K/k}{\log K} \right) + \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} \sum_{p | k} P \left(\frac{\log p}{\log K} \right) \tilde{f}_1 \left(\frac{\log K/k}{\log K} \right),$$

where f_1, \tilde{f}_1, P are some polynomials to be chosen later. The a_k given above are less general than the coefficients of Feng and Wu, but they are simpler, so we are able to write the Montgomery–Odlyzko expressions for our sequence explicitly.

2. Lemmas.

Lemma 1 (Mertens Theorem).

$$\sum_{p \leq y} \frac{\log p}{p} = \log y + O(1).$$

Lemma 2 (See Levinson [5]).

$$\sum_{p | j} \frac{\log p}{p} = O(\log \log j).$$

Lemma 3. For fixed $r \geq 1$,

$$\sum_{k \leq x} \frac{d_r(k)^2}{k} = A_r (\log x)^{r^2} + O \left((\log T)^{r^2-1} \right)$$

uniformly for $x \leq T$.

Lemma 4. Let a_i be integer for $1 \leq i \leq m$, $D > 1$ and f is a continuous function. Then

$$\begin{aligned} & \int_1^D \frac{\log^{a_1-1} x_1}{x_1} dx_1 \int_1^{\frac{D}{x_1}} \frac{\log^{a_2-1} x_2}{x_2} dx_2 \cdots \int_1^{\frac{D}{x_1 x_2 \cdots x_{m-1}}} \frac{f(x_1 x_2 \cdots x_m x)}{x} dx \\ &= \frac{\prod_{i=1}^m (a_i - 1)!}{(\sum_{i=1}^m a_i)!} \int_1^D \frac{f(x) \log^{\sum_{i=1}^m a_i} x}{x} dx. \end{aligned}$$

Lemma 5. Let a_i be integer for $1 \leq i \leq m$, and g is a polynomial. Then we have

$$\begin{aligned} & \sum_{k \leq K} \frac{d_r(k)^2}{k} g\left(\frac{\log K/k}{\log K}\right) \\ &= A_r r^2 \int_1^K g\left(\frac{\log K/x}{\log K}\right) (\log x)^{r^2-1} \frac{dx}{x} + O\left((\log K)^{r^2-1}\right) \end{aligned}$$

and

$$\begin{aligned} & \sum_{p_1 p_2 \cdots p_m \leq K} \prod_{i=1}^m \frac{\log^{a_i} p_i}{p_i} \mu^2(p_1 p_2 \cdots p_m) \sum_{k_0 \leq K/(p_1 p_2 \cdots p_m)} \frac{d_r(k_0)^2}{k_0} g\left(\frac{\log K/(p_1 p_2 \cdots p_m k_0)}{\log K}\right) \\ &= (1 + O(\log^{-1} K)) \\ & \times A_r r^2 \int_1^K \log^{a_1-1} x_1 \frac{dx_1}{x_1} \int_1^{\frac{K}{x_1}} \log^{a_2-1} x_2 \frac{dx_2}{x_2} \cdots \int_1^{\frac{K}{x_1 x_2 \cdots x_{m-1}}} \log^{a_m-1} x_m \frac{dx_m}{x_m} \\ & \times \int_1^{\frac{K}{x_1 x_2 \cdots x_m}} g\left(\frac{\log K/(x_1 x_2 \cdots x_m x)}{\log K}\right) (\log x)^{r^2-1} \frac{dx}{x}. \end{aligned}$$

For the proof of Lemma 5 using Lemmas 1–3, see [3].

3. Proof of Theorem 1. To give an upper bound for μ , we evaluate $h(c)$ in (1) with the coefficients

$$a_k = \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} f_1\left(\frac{\log K/k}{\log K}\right) + \frac{\lambda(k) d_r(k)}{k^{\frac{1}{2}}} \sum_{p|k} P\left(\frac{\log p}{\log K}\right) \tilde{f}_1\left(\frac{\log K/k}{\log K}\right),$$

where $r \geq 1$ and f_1, \tilde{f}_1, P are polynomials.

First, we evaluate the denominator in the ratio in the definition of $h(c)$.

$$\begin{aligned} \sum_{k \leq K} |a_k|^2 &= \sum_{k \leq K} \frac{d_r(k)^2}{k} f_1\left(\frac{\log K/k}{\log K}\right)^2 \\ &+ 2 \sum_{k \leq K} \frac{d_r(k)^2}{k} f_1\left(\frac{\log K/k}{\log K}\right) \tilde{f}_1\left(\frac{\log K/k}{\log K}\right) \sum_{p|k} P\left(\frac{\log p}{\log K}\right) \\ &+ \sum_{k \leq K} \frac{d_r(k)^2}{k} \tilde{f}_1\left(\frac{\log K/k}{\log K}\right)^2 \sum_{p|k} P\left(\frac{\log p}{\log K}\right) \sum_{q|k} P\left(\frac{\log q}{\log K}\right) \\ &= \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3. \end{aligned}$$

Using Lemma 5 and recalling that $K = T(\log T)^{-2}$, we have

$$\begin{aligned} \tilde{D}_1 &= A_r r^2 \int_1^K f_1\left(\frac{\log K/x}{\log K}\right)^2 (\log x)^{r^2-1} \frac{dx}{x} + O\left((\log T)^{r^2-1}\right) \\ &= A_r r^2 (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u)^2 du + O\left((\log T)^{r^2-1}\right) \\ &= A_r r^2 (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u)^2 du + O\left((\log T)^{r^2-1+\varepsilon}\right), \end{aligned} \tag{2}$$

where $\varepsilon > 0$ is arbitrarily small and the constant in the O -term depends on r , ε and f_1 . By Lemma 5 we obtain that

$$\begin{aligned}\tilde{D}_2 &= \frac{2A_r r^4}{\log K} \int_1^K \frac{P_1(\log x_1)}{x_1} \int_1^{\frac{K}{x_1}} f_1 \left(\frac{\log K/x_1 x}{\log K} \right) \\ &\quad \times \tilde{f}_1 \left(\frac{\log K/x_1 x}{\log K} \right) (\log x)^{r^2-1} \frac{dx}{x} dx_1 + O \left((\log T)^{r^2-1+\varepsilon} \right),\end{aligned}$$

where $P_1(y) = \frac{P(y)}{y}$. By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x}{\log K}$, we have

$$\begin{aligned}\tilde{D}_2 &= 2A_r r^4 (\log K)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v) \tilde{f}_1(v) dv du + O \left((\log T)^{r^2-1+\varepsilon} \right) \\ &= 2A_r r^4 (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v) \tilde{f}_1(v) dv du + O \left((\log T)^{r^2-1+\varepsilon} \right),\end{aligned}\tag{3}$$

where the constant in the O -term depends on r , ε and f_1, \tilde{f}_1 .

We have

$$\begin{aligned}\tilde{D}_3 &= \sum_{k \leq K} \frac{d_r(k)^2}{k} \tilde{f}_1 \left(\frac{\log K/k}{\log K} \right)^2 \sum_{p_1 p_2 | k} \mu^2(p_1 p_2) P \left(\frac{\log p_1}{\log K} \right) P \left(\frac{\log p_2}{\log K} \right) \\ &\quad + \sum_{k \leq K} \frac{d_r(k)^2}{k} \tilde{f}_1 \left(\frac{\log K/k}{\log K} \right)^2 \sum_{p|k} P^2 \left(\frac{\log p}{\log K} \right) \\ &= \tilde{D}_{31} + \tilde{D}_{32}.\end{aligned}$$

Again by Lemma 5 we obtain that

$$\begin{aligned}\tilde{D}_{31} &= \frac{A_r r^6}{\log^2 K} \int_1^K \frac{P_1(\log x_1)}{x_1} \int_1^{\frac{K}{x_1}} \frac{P_1(\log x_2)}{x_2} \int_1^{\frac{K}{x_1 x_2}} \tilde{f}_1 \left(\frac{\log K/x_1 x_2 x}{\log K} \right)^2 \\ &\quad \times (\log x)^{r^2-1} \frac{dx}{x} dx_2 dx_1 + O \left((\log T)^{r^2-1+\varepsilon} \right),\end{aligned}$$

where $P_1(y) = \frac{P(y)}{y}$. We remark that by Lemma 4 we can reduce the number of the repeated integrations in the above expression. By the change of variables $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_2}{\log K}$, $w = 1 - \frac{\log x_1 x_2 x}{\log K}$,

$$\begin{aligned}\tilde{D}_{31} &= A_r r^6 (\log K)^{r^2} \int_0^1 P_1(1-u) \int_{1-u}^1 P_1(1-v) \int_0^{u+v-1} (u+v-w-1)^{r^2-1} \tilde{f}_1(w)^2 dw dv du \\ &\quad + O \left((\log T)^{r^2-1+\varepsilon} \right) \\ &= A_r r^6 (\log T)^{r^2} \int_0^1 P_1(1-u) \int_{1-u}^1 P_1(1-v) \int_0^{u+v-1} (u+v-w-1)^{r^2-1} \tilde{f}_1(w)^2 dw dv du \\ &\quad + O \left((\log T)^{r^2-1+\varepsilon} \right),\end{aligned}\tag{4}$$

where the constant in the O -term depends on r , ε and \tilde{f}_1 . Similarly,

$$\begin{aligned}\tilde{D}_{32} &= A_r r^4 (\log K)^{r^2} \int_0^1 P_2(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v)^2 dv du + O \left((\log T)^{r^2-1+\varepsilon} \right) \\ &= A_r r^4 (\log T)^{r^2} \int_0^1 P_2(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v)^2 dv du + O \left((\log T)^{r^2-1+\varepsilon} \right),\end{aligned}\tag{5}$$

where $P_2(y) = \frac{P(y)^2}{y}$ and the constant in the O -term depends on r, ε and \tilde{f}_1 .

We now proceed to evaluation of the numerator in the ratio in (1). If we let

$$N(c) = \sum_{nk \leq K} a_k a_{nk} g_c(n) \Lambda(n) n^{-1/2},$$

then

$$\begin{aligned} N(c) = & \frac{2}{\pi} \sum_{nk \leq K} \frac{\lambda(k) d_r(k) \lambda(nk) d_r(nk) \Lambda(n)}{kn \log n} \sin \left(\pi c \frac{\log n}{\log T} \right) \times \left(f_1 \left(\frac{\log K/k}{\log K} \right) f_1 \left(\frac{\log K/nk}{\log K} \right) \right. \\ & + f_1 \left(\frac{\log K/nk}{\log K} \right) \tilde{f}_1 \left(\frac{\log K/k}{\log K} \right) \sum_{p_1|k} P \left(\frac{\log p_1}{\log K} \right) \\ & + f_1 \left(\frac{\log K/k}{\log K} \right) \tilde{f}_1 \left(\frac{\log K/nk}{\log K} \right) \sum_{p_1|nk} P \left(\frac{\log p_1}{\log K} \right) \\ & \left. + \tilde{f}_1 \left(\frac{\log K/k}{\log K} \right) \tilde{f}_1 \left(\frac{\log K/nk}{\log K} \right) \sum_{p_1|k} P \left(\frac{\log p_1}{\log K} \right) \sum_{q_1|nk} P \left(\frac{\log q_1}{\log K} \right) \right), \end{aligned}$$

so we can write

$$N(c) = N_1 + N_2 + N_3 + N_4.$$

Using the distribution of $\Lambda(n)$, we obtain

$$\begin{aligned} N_1 = & -\frac{2}{\pi} \sum_{pk \leq K} \frac{d_r(k) d_r(pk)}{kp} \sin \left(\pi c \frac{\log p}{\log T} \right) f_1 \left(\frac{\log K/k}{\log K} \right) f_1 \left(\frac{\log K/pk}{\log K} \right) + O \left((\log T)^{r^2-1} \right) \\ = & -\frac{2r}{\pi} \sum_{p \leq K} \frac{\sin \left(\pi c \frac{\log p}{\log T} \right)}{p} \sum_{k \leq K/p} \frac{d_r(k)^2}{k} f_1 \left(\frac{\log K/k}{\log K} \right) f_1 \left(\frac{\log K/pk}{\log K} \right) + O \left((\log T)^{r^2-1} \right). \end{aligned}$$

By Lemma 5 we have

$$\begin{aligned} N_1 = & -\frac{2A_r r^3}{\pi} \sum_{p \leq K} \frac{\sin \left(\pi c \frac{\log p}{\log T} \right)}{p} \int_1^{\frac{K}{p}} f_1 \left(\frac{\log K/x}{\log K} \right) f_1 \left(\frac{\log K/px}{\log K} \right) (\log x)^{r^2-1} \frac{dx}{x} \\ & + O \left((\log T)^{r^2-1} \right). \end{aligned}$$

From Lemma 1 and Abel's summation,

$$\begin{aligned} N_1 = & -\frac{2A_r r^3}{\pi} \int_1^K \frac{\sin \left(\pi c \frac{\log x_1}{\log T} \right)}{x_1 \log x_1} \int_1^{\frac{K}{x_1}} f_1 \left(\frac{\log K/x}{\log K} \right) f_1 \left(\frac{\log K/xx_1}{\log K} \right) (\log x)^{r^2-1} \frac{dx}{x} dx_1 \\ & + O \left((\log T)^{r^2-1} \right). \end{aligned}$$

Interchanging the order of integration and the names of the variables x and x_1 , we find

$$\begin{aligned} N_1 = & -\frac{2A_r r^3}{\pi} \int_1^K f_1 \left(\frac{\log K/x_1}{\log K} \right) \frac{(\log x_1)^{r^2-1}}{x_1} \int_1^{\frac{K}{x_1}} \frac{\sin \left(\pi c \frac{\log x}{\log T} \right)}{\log x} f_1 \left(\frac{\log K/xx_1}{\log K} \right) \frac{dx}{x} dx_1 \\ & + O \left((\log T)^{r^2-1} \right). \end{aligned}$$

Let $u = 1 - \frac{\log x_1}{\log K}$, $v = \frac{\log x}{\log K}$. Then

$$\begin{aligned}
N_1 &= -\frac{2A_r r^3}{\pi} (\log K)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u) \int_0^u \frac{\sin\left(\pi c v \frac{\log K}{\log T}\right)}{v} f_1(u-v) dv du \\
&\quad + O\left((\log T)^{r^2-1}\right) \\
&= -\frac{2A_r r^3}{\pi} (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u) \int_0^u \frac{\sin(\pi c v)}{v} f_1(u-v) dv du \\
&\quad + O\left((\log T)^{r^2-1+\varepsilon}\right),
\end{aligned} \tag{6}$$

where the constant in the O -term depends on r , ε and f_1 .

In N_2 we can replace the product of the summation variables nk by pp_1k_0 to get

$$\begin{aligned}
N_2 &= -\frac{2r^3}{\pi} \sum_{p_1 \leq K} \frac{P\left(\frac{\log p_1}{\log K}\right)}{p_1} \sum_{pk_0 \leq K/p_1} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right) d_r(k_0)^2}{pk_0} \\
&\quad \times f_1\left(\frac{\log K/(pp_1k_0)}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(p_1k_0)}{\log K}\right) + O\left((\log T)^{r^2-1}\right).
\end{aligned}$$

The inner sum $\sum_{pk_0 \leq K/p_1}$ in the expression above is the sum $\sum_{k_0 \leq K/p_1} \sum_{p \leq K/(p_1k_0)}$. As in the calculation of N_1 , we can show that this double sum is

$$\begin{aligned}
&A_r r^2 \int_1^{\frac{K}{p_1}} \tilde{f}_1\left(\frac{\log K/(p_1x_2)}{\log K}\right) \frac{(\log x_2)^{r^2-1}}{x_2} \\
&\times \int_1^{\frac{K}{p_1x_2}} \frac{\sin\left(\pi c \frac{\log x}{\log T}\right)}{\log x} f_1\left(\frac{\log K/(p_1xx_2)}{\log K}\right) \frac{dx}{x} dx_2 + O\left((\log T)^{r^2-1}\right).
\end{aligned}$$

By Lemma 1 we obtain

$$\begin{aligned}
N_2 &= -\frac{2A_r r^5}{\pi (\log K)} \int_1^K \frac{P_1\left(\frac{\log x_1}{\log K}\right)}{x_1} \int_1^{\frac{K}{x_1}} \tilde{f}_1\left(\frac{\log K/(x_1x_2)}{\log K}\right) \frac{(\log x_2)^{r^2-1}}{x_2} \\
&\times \int_1^{\frac{K}{x_1x_2}} \frac{\sin\left(\pi c \frac{\log x}{\log T}\right)}{\log x} f_1\left(\frac{\log K/(xx_1x_2)}{\log K}\right) \frac{dx}{x} dx_2 dx_1 + O\left((\log T)^{r^2-1}\right).
\end{aligned}$$

Making the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

$$\begin{aligned}
N_2 &= -\frac{2A_r r^5}{\pi} (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v) \int_0^v \frac{\sin(\pi cw)}{w} f_1(v-w) dw dv du \\
&\quad + O\left((\log T)^{r^2-1+\varepsilon}\right),
\end{aligned} \tag{7}$$

where the constant in the O -term depends on r , ε and f_1, \tilde{f}_1 .

As in N_1 and N_2 , the terms with $n = p$ for the primes p give the main contribution to N_3 :

$$\begin{aligned}
N_3 &= -\frac{2}{\pi} \sum_{pk \leq K} \sin\left(\pi c \frac{\log p}{\log T}\right) \frac{d_r(k) d_r(kp)}{kp} f_1\left(\frac{\log K/k}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right) \\
&\quad \times \sum_{p_1 | pk} P\left(\frac{\log p_1}{\log K}\right) + O\left((\log T)^{r^2-1}\right).
\end{aligned}$$

For $(p, k) = 1$ it follows that

$$\sum_{p_1|pk} P\left(\frac{\log p_1}{\log K}\right) = \sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right) + P\left(\frac{\log p}{\log K}\right). \quad (8)$$

Since the contribution of the terms with $(p, k) \neq 1$ in N_3 is $O\left((\log T)^{r^2-1}\right)$, then, according to decomposition (8), we can write

$$N_3 = N_{31} + N_{32} + O\left((\log T)^{r^2-1}\right),$$

where

$$\begin{aligned} N_{31} = & -\frac{2r^3}{\pi} \sum_{p_1 \leq K} \frac{P\left(\frac{\log p_1}{\log K}\right)}{p_1} \sum_{pk_0 \leq K/p_1} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right) d_r(k_0)^2}{pk_0} \\ & \times \tilde{f}_1\left(\frac{\log K/(pp_1k_0)}{\log K}\right) f_1\left(\frac{\log K/(p_1k_0)}{\log K}\right) + O\left((\log T)^{r^2-1}\right) \end{aligned}$$

and

$$\begin{aligned} N_{32} = & -\frac{2r}{\pi} \sum_{pk \leq K} \frac{\sin\left(\pi c \frac{\log p}{\log T}\right) d_r(k)^2 P\left(\frac{\log p}{\log K}\right)}{pk} \tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right) \\ & \times f_1\left(\frac{\log K/k}{\log K}\right) + O\left((\log T)^{r^2-1}\right). \end{aligned}$$

As in the calculation of N_2 we get

$$\begin{aligned} N_{31} = & -\frac{2A_r r^5}{\pi} (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v) \\ & \times \int_0^v \frac{\sin(\pi cw)}{w} \tilde{f}_1(v-w) dw dv du + O\left((\log T)^{r^2-1+\varepsilon}\right), \end{aligned}$$

and as in the calculation of N_1 ,

$$\begin{aligned} N_{32} = & -\frac{2A_r r^3}{\pi} (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u) \int_0^u \sin(\pi cv) P_1(v) \tilde{f}_1(u-v) dv du \\ & + O\left((\log T)^{r^2-1+\varepsilon}\right), \end{aligned}$$

where $P_1(y) = \frac{P(y)}{y}$. Thus,

$$\begin{aligned} N_3 = & -\frac{2A_r r^5}{\pi} (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} f_1(v) \\ & \times \int_0^v \frac{\sin(\pi cw)}{w} \tilde{f}_1(v-w) dw dv du \\ & -\frac{2A_r r^3}{\pi} (\log T)^{r^2} \int_0^1 (1-u)^{r^2-1} f_1(u) \int_0^u \sin(\pi cv) P_1(v) \tilde{f}_1(u-v) dv du \\ & + O\left((\log T)^{r^2-1+\varepsilon}\right), \end{aligned} \quad (9)$$

where the constant in the O -term depends on r, ε and f_1, \tilde{f}_1, P .

Again, in the sum defining N_4 we can replace the integers $n \geq 2$ with the primes p :

$$\begin{aligned} N_4 = & -\frac{2}{\pi} \sum_{pk \leq K} \sin\left(\pi c \frac{\log p}{\log T}\right) \frac{d_r(k) d_r(kp)}{kp} \tilde{f}_1\left(\frac{\log K/k}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(pk)}{\log K}\right) \\ & \times \sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right) \sum_{q_1|pk} P\left(\frac{\log q_1}{\log K}\right) + O\left((\log T)^{r^2-1}\right). \end{aligned}$$

For the two innermost sums, if $(k, p) = 1$, we have

$$\begin{aligned} & \sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right) \sum_{q_1|pk} P\left(\frac{\log q_1}{\log K}\right) \\ = & \sum_{p_1 q_1|k} \mu^2(p_1 q_1) P\left(\frac{\log p_1}{\log K}\right) P\left(\frac{\log q_1}{\log K}\right) \\ & + \sum_{p_1|k} P^2\left(\frac{\log p_1}{\log K}\right) + P\left(\frac{\log p}{\log K}\right) \sum_{p_1|k} P\left(\frac{\log p_1}{\log K}\right). \end{aligned}$$

According to this decomposition, we write

$$N_4 = N_{41} + N_{42} + N_{43}.$$

As before, by Lemma 5 we find

$$\begin{aligned} N_{41} = & -\frac{2A_r r^7}{\pi(\log K)^2} \int_1^K \frac{P_1\left(\frac{\log x_1}{\log K}\right)}{x_1} \int_1^{\frac{K}{x_1}} \frac{P_1\left(\frac{\log x_2}{\log K}\right)}{x_2} \int_1^{\frac{K}{x_1 x_2}} \tilde{f}_1\left(\frac{\log K/(x_1 x_2 x_3)}{\log K}\right) \frac{(\log x_3)^{r^2-1}}{x_3} \\ & \times \int_1^{\frac{K}{x_1 x_2 x_3}} \frac{\sin\left(\pi c \frac{\log x}{\log T}\right)}{\log x} \tilde{f}_1\left(\frac{\log K/(x x_1 x_2 x_3)}{\log K}\right) \frac{dx}{x} dx_3 dx_2 dx_1 + O\left((\log T)^{r^2-1}\right). \end{aligned}$$

Making the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_2}{\log K}$, $w = 1 - \frac{\log x_1 x_2 x_3}{\log K}$, $z = \frac{\log x}{\log K}$, we get

$$\begin{aligned} N_{41} = & -\frac{2A_r r^7}{\pi} (\log T)^{r^2} \int_0^1 P_1(1-u) \int_{1-u}^1 P_1(1-v) \\ & \times \int_0^{u+v-1} (u+v-w-1)^{r^2-1} \tilde{f}_1(w) \int_0^w \frac{\sin(\pi cz)}{z} \tilde{f}_1(w-z) dz dw dv du \\ & + O\left((\log T)^{r^2-1+\varepsilon}\right), \end{aligned} \tag{10}$$

where the constant in the O -term depends on r, ε and f_1, \tilde{f}_1, P .

Next,

$$\begin{aligned} N_{42} = & -\frac{2A_r r^5}{\pi(\log K)} \int_1^K \frac{P_2\left(\frac{\log x_1}{\log K}\right)}{x_1} \int_1^{\frac{K}{x_1}} \tilde{f}_1\left(\frac{\log K/(x_1 x_2)}{\log K}\right) \frac{(\log x_2)^{r^2-1}}{x_2} \\ & \times \int_1^{\frac{K}{x_1 x_2}} \frac{\sin\left(\pi c \frac{\log x}{\log T}\right)}{\log x} \tilde{f}_1\left(\frac{\log K/(x x_1 x_2)}{\log K}\right) \frac{dx}{x} dx_2 dx_1 + O\left((\log T)^{r^2-1}\right). \end{aligned}$$

Degrees			Value of c Value of r		Polynomials		
f_1	f_1	P			f_1	f_1	P
3	1	2	0.515398	1.18	$1.95 + 1.47x - 1.07x^2 - 0.29x^3$	$-0.7 - 1.92x$	x^2
3	1	3	0.515397	1.18	$1.655 + 1.25x - 0.886x^2 - 0.25x^3$	$-0.57 - 1.6x$	$x^2 + 0.036x^3$
6	2	3	0.515396	1.18	$1.78 + 1.017x + 0.2x^2 - 1.56x^3 + 0.45x^4 - 0.06x^5 + 0.05x^6$	$-0.629 - 0.88x - 1.799x^2$	$x^2 + 0.083x^3$

Table 1: Numerically optimal polynomials in the coefficients $\{a_k\}$, for which $h(c) > 1$.

By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

$$N_{42} = -\frac{2A_r r^5}{\pi} (\log T)^{r^2} \int_0^1 P_2(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v) \int_0^v \frac{\sin(\pi cw)}{w} \tilde{f}_1(v-w) dw dv du + O\left((\log T)^{r^2-1+\varepsilon}\right), \quad (11)$$

where the constant in the O -term depends on r , ε and \tilde{f}_1 , P .

Finally,

$$N_{43} = -\frac{2A_r r^5}{\pi (\log K)^2} \int_1^K \frac{P_1\left(\frac{\log x_1}{\log K}\right)}{x_1} \int_1^{\frac{K}{x_1}} \tilde{f}_1\left(\frac{\log K/(x_1 x_2)}{\log K}\right) \frac{(\log x_2)^{r^2-1}}{x_2} \times \int_1^{\frac{K}{x_1 x_2}} \sin\left(\pi c \frac{\log x}{\log T}\right) P_1\left(\frac{\log x}{\log K}\right) \tilde{f}_1\left(\frac{\log K/(x x_1 x_2)}{\log K}\right) \frac{dx}{x} dx_2 dx_1 + O\left((\log T)^{r^2-1}\right).$$

By the variable changes $u = 1 - \frac{\log x_1}{\log K}$, $v = 1 - \frac{\log x_1 x_2}{\log K}$, $w = \frac{\log x}{\log K}$, we get

$$N_{43} = -\frac{2A_r r^5}{\pi} (\log T)^{r^2} \int_0^1 P_1(1-u) \int_0^u (u-v)^{r^2-1} \tilde{f}_1(v) \int_0^v \sin(\pi cw) P_1(w) \tilde{f}_1(v-w) dw dv du + O\left((\log T)^{r^2-1+\varepsilon}\right), \quad (12)$$

where the constant in the O -term depends on r , ε and \tilde{f}_1 , P .

Using D_i , N_i given by (2)–(12) we can evaluate

$$h(c) = c - \frac{N_1 + N_2 + N_3 + N_4}{D_1 + D_2 + D_3}.$$

The results of our numerical calculations are summarized in Table 1.

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